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CRITERION OF STABILITY OF LINEAR SYSTEMS WITH VARIABLE COEFFICIENTS AND TIME LAG

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A criterion of stability given by [1] for the linear systems of ordinary differential equations with variable coefficients, is extended to systems with lag. The explicit form of the quadratic functionals developed in [2] for the systems of differential equations with lag is used. These functionals play a role analogous to that of the Liapunov quadratic forms for the systems of ordinary differential equations. A criterion of stability of a nonlinear system of differential equations with lag is obtained in the manner analogous to that of [1].

Let the system

$$dx/dt = Ax(t) + Bx(t-\tau), \quad \tau = \text{const}, \ \tau > 0 \tag{1}$$

where A and B are $n \times n$ constant matrices and $x(t)$ is an n-dimensional vector be asymptotically stable, i.e., let the roots of the characteristic equation

$$|A - \lambda E + B\exp(-\lambda\tau)| = 0 \tag{2}$$

have negative real parts. Then by Theorem 5.1 of [2], positive definite quadratic functionals $V[x(\vartheta)]$ and $W[x(\vartheta)]$ exist and $V[x(\vartheta)]$ has the form

$$V[\mathbf{x}(\vartheta)] = (\alpha\mathbf{x}(0)\cdot\mathbf{x}(0)) + \int_{-\tau}^{0} (\beta(\vartheta)\,\mathbf{x}(\vartheta)\cdot\mathbf{x}(0))\,d\vartheta + \int_{-\tau}^{0}\int_{-\tau}^{0} (\gamma(\vartheta,\xi)\,\mathbf{x}(\vartheta)\cdot\mathbf{x}(\xi))\,d\vartheta d\xi \tag{3}$$

Here $\alpha = \{\alpha_{ij}\}$, $\alpha_{ij} = \alpha_{ji}$, $\beta(\vartheta) = \{\beta_{ij}(\vartheta)\}$, $\gamma(\vartheta,\xi) = \{\gamma_{ij}(\vartheta,\xi)\}$, $\gamma_{ij}(\vartheta,\xi) = \gamma_{ji}(\xi,\vartheta)$, $(i, j = 1, 2, \ldots n)$, α_{ij} are constants, $\beta_{ij}(\vartheta)$ and $\gamma_{ij}(\vartheta,\xi)$ are continuously differentiable functions and $(\mathbf{x}\cdot\mathbf{y})$ is a scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$.

The total differentiable of the functional $V[\mathbf{x}(\vartheta)]$ with respect to time satisfies, by virtue of the system (1), the condition

$$\left.\frac{dV[\mathbf{x}_t(\vartheta)]}{dt}\right|_{(1)} = -W[\mathbf{x}(\vartheta)], \qquad \mathbf{x}_t(\vartheta) = \mathbf{x}(t+\vartheta), \qquad -\tau \leqslant \vartheta \leqslant 0 \tag{4}$$

where $\mathbf{x}_t(\vartheta)$ represents an element of the trajectory of the system (1). Let us define on the functional space of continuous functions

$$\mathbf{x}(\vartheta) = \{x_i(\vartheta)\} \quad (i = 1, 2, \ldots n, \ -\tau \leqslant \vartheta \leqslant 0)$$

the norm

$$\|\mathbf{x}(\vartheta)\|_\tau = \sup(\mathbf{x}(\vartheta)\cdot\mathbf{x}(\vartheta))^{1/2} \ (-\tau \leqslant \vartheta \leqslant 0)$$

and consider the functionals $V[\mathbf{x}(\vartheta)]$ and $W[\mathbf{x}(\vartheta)]$ on the hypersphere

$$\| x(\vartheta) \|_\tau^2 = 1 \tag{5}$$

Limiting ourselves to the functions $x(\vartheta)$ satisfying the Lipshitz conditions, i.e., to the functions defined on the interval $[-\tau, 0]$ for which a constant K exist for any $\vartheta', \vartheta'' \in [-\tau, 0]$ such that

$$| x_i(\vartheta'') - x_i(\vartheta') | < K | \vartheta'' - \vartheta' | \qquad (i = 1, 2, \ldots n)$$

we find that the set of functions $x(\vartheta)$ satisfying the condition (5) will be compact. Consequently the functionals $V[x(\vartheta)]$ and $W[x(\vartheta)]$ are bounded on this set and attain the strict lower and upper bound. Since the functionals $V[x(\vartheta)]$ and $W[x(\vartheta)]$ are positive definite, positive numbers l, l_1, L and L_1 exist, such that the following equalities hold on the functional hypersphere (5):

$$\begin{aligned} \inf V[x(\vartheta)] &= l, & \sup V[x(\vartheta)] &= L \\ \inf W[x(\vartheta)] &= l_1, & \sup W[x(\vartheta)] &= L_1 \end{aligned} \tag{6}$$

As $V[x(\vartheta)]$ and $W[x(\vartheta)]$ are quadratic functionals, the following inequalities hold:

$$\begin{aligned} l \| x(\vartheta) \|_\tau^2 &\leqslant V[x(\vartheta)] \leqslant L \| x(\vartheta) \|_\tau^2 \\ l_1 \| x(\vartheta) \|_\tau^2 &\leqslant W[x(\vartheta)] \leqslant L_1 \| x(\vartheta) \|_\tau^2 \end{aligned} \tag{7}$$

We can regard the set of functions $x(\vartheta)$ as a set of initial functions for solving the systems of differential equations with lag without any loss of generality, by virtue of the note appearing in [3] p. 158. This means that if for a given t_0 the system is asymptotically stable on the set of initial functions, for which the initial instant is defined by $t_0 + \tau$ and which satisfy the Lipshitz conditions, then the system is also asymptotically stable on a set of arbitrary, piecewise continuous curves $x_{t_0}(\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$ for which the initial instant is t_0.

Consider the system

$$dx/dt = (A + C(t))\, x(t) + (B + D(t))\, x(t - \tau) \tag{8}$$

where $C(t)$ and $D(t)$ are $n \times n$ matrices continuous in t. Let us find the derivative of the functional (3) with respect to time, using the system (8)

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(8)} = &- W[x(\vartheta)] + ((\alpha C(t) + C^*(t)\alpha)\, x(0) \cdot x(0)) + (2\alpha D(t)\, x(-\tau) \cdot x(0)) + \\ &+ \int_{-\tau}^{0} (C^*(t)\, \beta(\vartheta)\, x(\vartheta) \cdot x(0))\, d\vartheta + \int_{-\tau}^{0} (D^*(t)\, \beta(\vartheta)\, x(\vartheta) \cdot x(-\tau))\, d\vartheta \end{aligned} \tag{9}$$

The inequalities (7) yield the following estimate:

$$W[x(\vartheta)] \geqslant l_1 \| x(\vartheta) \|_\tau^2 \geqslant \frac{l_1}{L} V[x(\vartheta)]$$

Next we consider the bilinear functional

$$P(x(\vartheta), x(0)) = \int_{-\tau}^{0} (\beta(\vartheta)\, x(\vartheta) \cdot x(0))\, d\vartheta = (U(x(\vartheta)) \cdot x(0))$$

$$U(x(\vartheta)) = \int_{-\tau}^{0} \beta(\vartheta)\, x(\vartheta)\, d\vartheta$$

where $\beta(\vartheta)$ is a continuously differentiable matrix, i.e. $U(x(\vartheta))$ is a linear operator.

The following relation holds for the norm of a bilinear functional [4] of the given type

$$\| P \| = \sup \| P(\mathrm{x}(\vartheta), \quad \mathrm{x}(0)) \| = \| U \| \text{ when } \| \mathrm{x}(\vartheta) \|_{\tau}, \quad \| \mathrm{x}(0) \| \leqslant 1 \tag{10}$$

and in particular we have

$$\sup (\alpha \mathrm{x}(0) \cdot \mathrm{x}(0)) = \| \alpha \| \quad \text{when } \| \mathrm{x}(0) \| \leqslant 1 \tag{11}$$

Relations (10) and (11) together yield

$$\| \alpha \| + \| U \| \leqslant L \tag{12}$$

The inequalities (12) and (7) give the following estimate:

$$\left. \frac{dV}{dt} \right|_{(8)} \leqslant -\frac{l_1}{L} V[\mathrm{x}(\vartheta)] + (\| C(t) \| + \| D(t) \|) \frac{2L}{l} V[\mathrm{x}(\vartheta)]$$

Integrating this inequality we find that the following relation holds along the segment of the trajectory of (8):

$$V[\mathrm{x}_t(\vartheta)] \leqslant V[\mathrm{x}_{t_0}(\vartheta)] \exp \left[-\frac{l_1}{L} + \frac{2L}{l(t-t_0)} \int_{t_0}^{t} (\| C(s) \| + \| D(s) \|) \, ds \right] (t - t_0)$$

Obviously, if

$$\lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} (\| C(s) \| + \| D(s) \|) \, ds < \frac{l_1 l}{2L^2} \tag{13}$$

then

$$\lim_{t \to \infty} V[\mathrm{x}_t(\vartheta)] = 0$$

and also

$$\lim_{t \to \infty} \mathrm{x}(t) = 0$$

Since the condition that the solutions of a linear system with lag tend to zero as $t \to \infty$ represents the sufficient condition for the asymptotic stability [5], the following theorem is true.

Theorem 1. If the system (1) is asymptotically stable, then so is the system (8) provided that the matrices $C(t)$ and $D(t)$ satisfy the condition (13) in which $l = \inf V[\mathrm{x}(\vartheta)]$, $L = \sup V[\mathrm{x}(\vartheta)]$ and $l_1 = \inf W[\mathrm{x}(\vartheta)]$ on the hypersphere $\| \mathrm{x}(\vartheta) \|_{\tau}^{2} = 1$ of the functions $\mathrm{x}(\vartheta) = \{x_i(\vartheta)\}$ $(i = 1, 2, \ldots n)$, satisfying the Lipshitz conditions, while $V[\mathrm{x}(\vartheta)]$ and $W[\mathrm{x}(\vartheta)]$ are quadratic functionals which represent the solution of the problem of asymptotic stability of (1) and satisfy the condition (4).

Clearly, when lag is absent, the theorem reduces to the corresponding theorem of [1]. The corresponding corollaries can be derived in a similar manner.

Corollaries. If the system (1) is asymptotically stable, so is the system (8) provided that any one of the following relations holds:

$$\begin{aligned}
&1. \int_{t_0}^{\infty} (\| C(s) \| + \| D(s) \|) \, ds < c < \infty \\
&2. \| C(t) \| + \| D(t) \| < {}^1\!/_2 l_1 L^{-1} \\
&3. \lim_{t \to \infty} (\| C(t) \| + \| D(t) \|) < {}^1\!/_2 l_1 l L^{-2} \\
&4. \lim_{t \to \infty} (\| C(t) \| + \| D(t) \|) = 0
\end{aligned} \tag{14}$$

Let us in addition consider the case of a nonlinear system

$$dx/dt = (A + C(t))\,x(t) + (B + D(t))\,x(t-\tau) + R(x(t), x(t-\tau), t) \tag{15}$$

where $R(x(t), x(t-\tau), t)$ is an n-dimensional vector continuous in all its arguments.

Theorem 2. If the system (1) is asymptotically stable and the matrices $C(t)$ and $D(t)$ satisfy the condition (14), then a constant $\beta > 0$ can be found, such that the trivial solution of (15) will be asymptotically stable for any value of the continuous vector $R(x, y, t)$ satisfying the inequality

$$\| R(x, y, t) \| \leqslant \beta (\| x \| + \| y \|) \tag{16}$$

Here l_1 and L are given by (6), while $V[x(\vartheta)]$ and $W[x(\vartheta)]$ are quadratic functionals which solve the problem of asymptotic stability of (1), and are such that

$$\left.\frac{dV[x_t(\vartheta)]}{dt}\right|_{(1)} = -2W[x(\vartheta)] \tag{17}$$

Proof. Let us compute a derivative with respect to time of the functional $V[x(\vartheta)]$ given by (3) and satisfying the condition (17), according to (15)

$$\left.\frac{dV}{dt}\right|_{(15)} = -2W[x(\vartheta)] + ((\alpha C(t) + C^*(t)\alpha)\,x(0)\cdot x(0)) + (2\alpha D(t)\,x(-\tau)\cdot x(0)) +$$
$$+ \int_{-\tau}^{0} (C^*(t)\,\beta(\vartheta)\,x(\vartheta)\cdot x(0))\,d\vartheta + \int_{-\tau}^{0} (D^*(t)\beta(\vartheta)\,x(\vartheta)\cdot x(-\tau))\,d\vartheta +$$
$$+ (\alpha R(x(0), x(-\tau), t) + R^*(x(0), x(-\tau), t)\,\alpha\cdot x(0)) +$$
$$+ \Big(\int_{-\tau}^{0} \beta(\vartheta)\,x(\vartheta)\,d\vartheta\cdot R(x(0), x(-\tau), t)\Big)$$

Making use of the estimate

$$(\alpha R(x(0), x(-\tau), t) + R^*(x(0), x(-\tau), t)\,\alpha\cdot x(0)) \leqslant$$
$$\leqslant 2\|\alpha\|\cdot R(x(0), x(-\tau), t)\|\cdot\|x(0)\| \leqslant 4\|\alpha\|\cdot\beta\cdot\|x(\vartheta)\|_\tau^2$$
$$\Big(\int_{-\tau}^{0} \beta(\vartheta)\,x(\vartheta)\,d\vartheta\cdot R(x(0), x(-\tau)\,t)\Big) \leqslant$$
$$\leqslant \Big\| \int_{-\tau}^{0} \beta(\vartheta)\,x(\vartheta)\,d\vartheta \Big\| \cdot \| R(x(0), x(-\tau),\ t)\| \leqslant 2\beta\|U\|\cdot\|x(\vartheta)\|_\tau^2$$

we obtain the following expression for the derivative of the functional:

$$\left.\frac{dV}{dt}\right|_{(15)} \leqslant -2W[x(\vartheta)] + (\|C(t)\| + \|D(t)\|)\,2L\,\|x(\vartheta)\|_\tau^2 + 4L\beta\,\|x(\vartheta)\|_\tau^2$$

If the sum $\|C(t)\| + \|D(t)\|$ satisfies the inequality (14) and the number $\beta > 0$ satisfies the condition $\beta < {}^1/_4 l_1 L^{-1}$, then the above derivative will be a negative definite functional, i.e. the functional $V[x(\vartheta)]$ will satisfy all conditions of the theorem on the asymptotic stability [3] along the trajectories of (15). Therefore the trivial solution of (15) is asymptotically stable, QED.

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MEAN SQUARE STABILITY OF LINEAR SYSTEMS UNDER THE ACTION OF A MARKOV CHAIN

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The problem of the mean square stability of a linear system which is under the action of a Markov chain is reduced to the investigation of the stability of the system for the second moments from the solutions of the original system. The system for the second moments possesses the property that its solutions, corresponding in a specific sense to positive initial data, are positive. This property permits us to apply to the investigation of the stability problem the very well developed theory of positive operators in a linear space with a cone.

1. Equations for second moments. We consider a system of n linear differential equations

$$dx/dt = A(u)\,x \tag{1.1}$$

which is under the action of a homogeneous Markov chain $\{u(t),\ 0 \leqslant t < \infty\}$ with a finite number of states [1, 2]. The behavior of the Markov chain is described by the transition probabilities $p_{ij}(t) = P(t,\ u_i,\ \{u_j\})$; here the matrix $P(t) = \{p_{ij}(t)\}$ satisfies the equality $P(t) = e^{Qt}$, where Q is an infinitesimal matrix with elements

$$q_{ij} = \begin{cases} \lim\limits_{t\to 0} t^{-1} p_{ij}(t), & j \neq i \\ \lim\limits_{t\to 0} t^{-1}(p_{ii}(t) - 1), & j = i \end{cases}$$

We introduce the numbers $q_i = -q_{ii}$ $(i = 1, \ldots, N)$ and the matrices $A_k = A(u_k)$ $(k = 1, \ldots, N)$. The Markov process generated by system (1.1) is denoted, as in [2], by $\{x(t),\ u(t),\ 0 \leqslant t < \infty\}$. The solution of system (1.1), corresponding to the initial data $x(0) = x^\circ$, $u(0) = u_k$, is written in the form $x(t;\ x^\circ,\ u_k)$. By the norm of a vector x we mean its Euclidean norm $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$

Definition (see [1]). The trivial solution of system (1.1) is said to be asymptotically mean square stable if for any number $\varepsilon > 0$ we can find a number $\delta > 0$ such